14. I. I. Mezhirov and Yu. I. Chistov, "Investigation of total pressure recovery in the flow downstream of supersonic nozzles," Uchen. Zap. TsAGI, 12, No. 2 (1981).
15. Yu. A. Tikhomirov, "Investigation of the effect of the Reynolds number on the pressure recovery coefficient in the diffuser of a supersonic wind tunnel," Uchen. Zap. TsAGI, 8, No. 2 (1977).
16. V. G. Gurylev and A. K. Trifonov, "Pseudoshock in a very simple air inlet in the form of a cylindrical pipe," Uchen. Zap. TsAGI, 7, No. 2 (1977).

DEVELOPMENT OF INTERNAL WAVES GENERATED BY A CONCENTRATED PULSE SOURCE IN AN INFINITE UNIFORMLY STRATIFIED FLUID

N. A. Zavol'skii and A. A. Zaitsev

UDC 532.592+551.466.81

1. The question of the fundamental function of the internal wave operator (FFIWO) in an infinite incompressible uniformly stratified fluid has been raised several times in recent years (see [1-4], say). This is natural: Its solution is of top priority in construction of a theory of linear forced internal waves (FIW). Almost everything is now known about FFIWO; in particular, a representation has been found in the form of a single integral in the frequencies, and its asymptotic has been obtained for large values of the time. Available information about the field of unsteady FIW is not detailed enough, however; there is no information about the limits of applicability of the asymptotic formulas, and no complete clarity relative to the initial stage of wave field evolution. The gap can be eliminated by using a numerical computation, but this has not yet been done, and this paper intends to fill this gap. Its main content is the elucidation of results of a computer computation of the integral representation for an unsteady FIW field, and their comparison with a computation of the asymptotic for this field in order to determine its range of action.

The FFIWO has no direct physical meaning; hence, the main attention will not be turned to this function. From the physical viewpoint, the consideration of the wave function of a concentrated pulse source (CPSF) is expedient (especially if we have in mind the application to problems of stratified fluid flow around bulk bodies). It was not apparently studied in detail earlier. Our computations refer precisely to the case of the action of a concentrated pulse source in an infinite incompressible uniformly stratified fluid. Since both functions, the FFIWO and CPSF, are closely interrelated, appropriate deductions relative to the properties and behavior of the fundamental function can be made from these computations. There are certainly distinctions. They will be mentioned below, together with the similarity in the behavior of these functions.

2. As the CPSF we take the vertical displacement of a fluid particle due to the FIW. We denote it by $\zeta = \zeta(x, y, z, t)$. Here x, y, z are the two horizontal and one vertical coordinate of the observation point (the z axis is directed upward), and t is the time. The origin of the right-handed rectangular coordinate system is set at the location of the pulse source whose action occurs instantaneously at the time $t = 0$ and causes the appearance of the FIW. We consider that an incompressible fluid fills the whole space and is stratified uniformly along the vertical. The Weisailla frequency therein is constant and equal to N. The examination is performed within the framework of linear theory and the Boussinesq approximation.

The equation of FIW dynamics has the following form in this case

$$\left(\partial_t^2\left(\partial_x^2+\partial_y^2+\partial_z^2\right)+N^2\left(\partial_x^2+\partial_y^2\right)\right)\zeta(x, y, z, t)=Q\partial_z\partial_t\delta(x, y, z, t), \tag{2.1}$$

where Q is the total debit of the source of the mass, ∂_x, ∂_y, ∂_z, ∂_t are differentiation operators with respect to x, y, z, t, respectively, and $\delta(x, y, z, t)$ is the Dirac function. The quantity ζ satisfies the causality condition

$$\zeta=0 \quad \text{for} \quad t<0 \tag{2.2}$$

(in the sense of the theory of generalized functions [5]).

Gor'kii. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnischekoi Fiziki, No. 6, pp. 58-64, November-December, 1984. Original article submitted July 24, 1983.

 0021-8944/84/2506-0862$08.50

The solution of the problem (2.1), (2.2) is unique (this is proved easily by using the theories of the Laplace and Fourier transforms of generalized functions [5]. Let us note that this is a more natural and easy path to the proof of the uniqueness theorem as compared with that proposed in [6, 7], and the CPSF is expressed in terms of the FFIWO in the form $\zeta = Q\partial_z\partial_t e$, where we use the notation $e = e(x, y, z, t)$.

The integral representation [2-4]

$$e = -H(t)/2\pi^2 R \cdot \int_n^N ((N^2-\omega^2)(\omega^2-n^2))^{-1/2} \sin \omega t d\omega, \tag{2.3}$$

holds for the FFIWO, where $n = N|z|/R$, $R = (r^2 + z^2)^{1/2}$, $r = (x^2 + y^2)^{1/2}$, and $H(t)$ is the Heaviside function.

Formal differentiation of the right side of (2.3) with respect to z results in an integral that diverges at the point $\omega = n$. It is regularized as follows. First the expression $(\partial_t^2 + n^2)\zeta$, which has a representation in the form of a regular integral, is calculated, and then its convolution is taken with the fundamental function of the operator $\partial_t^2 + n^2$. This yields

$$\zeta = QH(t)z/2\pi^2R^3 \cdot \left[0.5\pi \cos nt + \int_n^N \omega (N^2-\omega^2)^{1/2}(\omega^2-n^2)^{-3/2}(\cos nt - \cos \omega t)\, d\omega\right]. \tag{2.4}$$

For a numerical computation of the function ζ, replacement of the variable of integration

$$\omega = (n^2\cos^2\gamma + N^2\sin^2\gamma)^{1/2},$$

is useful and results in the following integral representation

$$\zeta = Qz/2\pi^2R^3 \cdot \int_0^{\pi/2} (\cos nt - \cos^2\gamma \cdot \cos \omega t)\, d\gamma/\sin^2\gamma. \tag{2.5}$$

It is easy to confirm that the singularity in the integrand at $\gamma = 0$ is eliminable.

The asymptotic of the FFIWO defined by (2.3) on the basis of formulas in [8] has the form

$$e \simeq -(2\pi)^{-3/2}R^{-1}(nt)^{-1/2}(N^2-n^2)^{-1/2}(n^{-1}\sin(nt+\pi/4) + N^{-1}\sin(Nt-\pi/4)),\quad nt \to \infty,\quad N-n > \varepsilon > 0. \tag{2.6}$$

The asymptotic formula for e can be differentiated [8]. Consequently, the following asymptotic is obtained for vertical displacements of fluid particles due to the FIW

$$\zeta \simeq Q(2\pi)^{-3/2}zR^{-3}n^{-1/2}(t(N^2-n^2))^{1/2}\cdot \sin(nt+\pi/4),\quad nt \to \infty,\quad N-n > \varepsilon > 0. \tag{2.7}$$

The condition $N - n > \varepsilon > 0$, hindering the closure of the boundaries of the integration range in (2.3), makes (2.6) and (2.7) inapplicable in a small vertical two-sheeted cone with apex at the point of source action. On the vertical axis the integral (2.5) is calculated directly and the result has the form

$$\zeta = Q \operatorname{sign} z \cdot \cos Nt/4\pi |z|^2,\quad t > 0. \tag{2.8}$$

3. Integral representations for the CPSF and FFIWO are expansions in harmonics with a finite and identical frequency range for both functions, where the lower boundary was the local frequency $n > 0$ and the upper was the Weisalla frequency. The frequency distribution of these functions has the following qualitative distinction: The spectral density of the CPSF vanishes at the uper boundary of the frequency band while the spectral density of the FFIWO is infinite there. Both densities grow to infinite in the neighborhood of the lower boundary of the range of integration, but the former grows substantially more rapidly. This predetermines the difference in the behaviors of both functions in space and in time upon retention of the similarity of a number of the fundamental features of the wave field pattern.

At the initial instant the FFIWO field is zero, then it grows and later decomposes asymptotically into the sum of two waves, one of which is propagated while the other performs standing oscillations at the Waisalla frequency. After these waves have been formed, their amplitudes start to decrease in proportion to $t^{-1/2}$ as time passes.

The field of unsteady FIW generated by the source evolves as follows. At the time of source action, its perturbation is transmitted at once to all points of the liquid medium be-

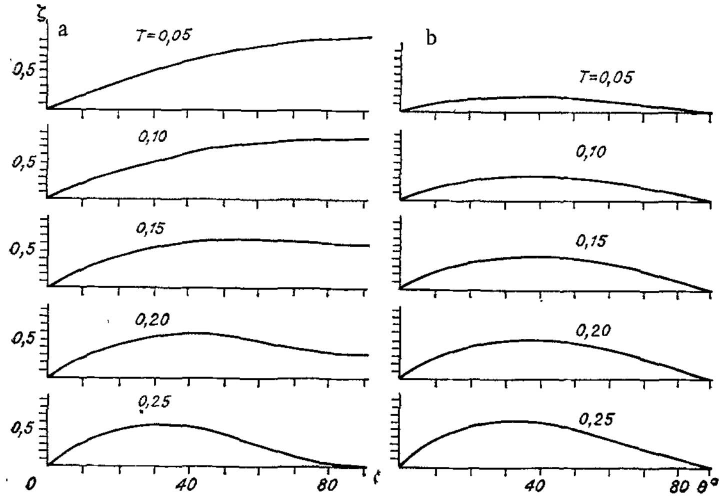

Fig. 1

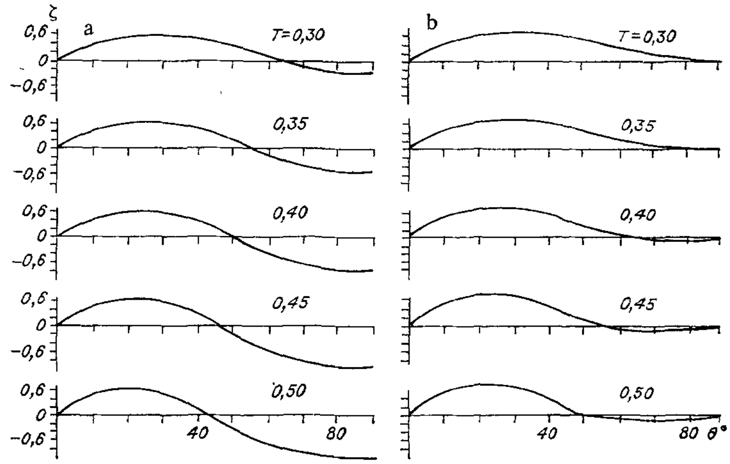

Fig. 2

cause of its incompressibility. The fluid particles leave their initial locations and shift instantaneously to new sites. The magnitude of the vertical displacement of the liquid particles at the initial time is determined by the equality

$$\zeta|_{t=0} = Qz/4\pi R^3.$$

Consequently, the water medium acquires a potential energy that starts to be transformed into FIW kinetic energy here. In contrast to the FFIWO, splitting of the field of unsteady waves generated by the concentrated source does not occur with the lapse of time. There are no standing waves oscillating at all points of the fluid at the Waisalla frequency in this case. The exception is points of the vertical axis passing through the source. The value of the vertical displacement is determined by (2.8) there. At the remaining points of the fluid, just the field of the waves being propagated is formed, where their amplitude does not decrease with time but grows in proportion to $t^{1/2}$.

The fact of the pulse-source-generated FIW growth with time has apparently not been discussed earlier. However, it should be kept in mind, especially in an examination of the problem of determining the field of stationary waves that occur in a stratified fluid flow around a bulk body. Attempts are often made to determine this field by making the solution of the unsteady problem of the flow around a system of sources and sinks stationary, but as we see, it is not made stationary. The output will possibly be found by replacing the singular sources and sinks by distributed ones, but such problems are substantially more difficult of solution and are not examined.

The FIW constant phase surfaces have the form of vertical cones with apices at the points of source action. The wave amplitude in the vertical particle displacements diminishes in inverse proportion to the square of the distance to the source. It is convenient to study the FIW field by considering its projection on a sphere of unit radius with center at the point of

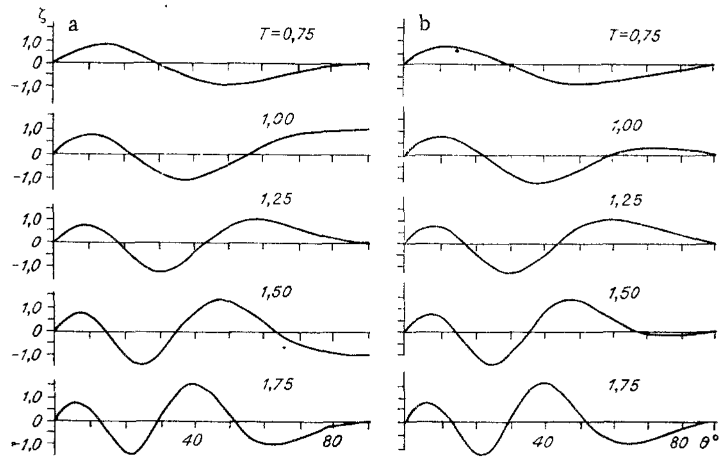

Fig. 3

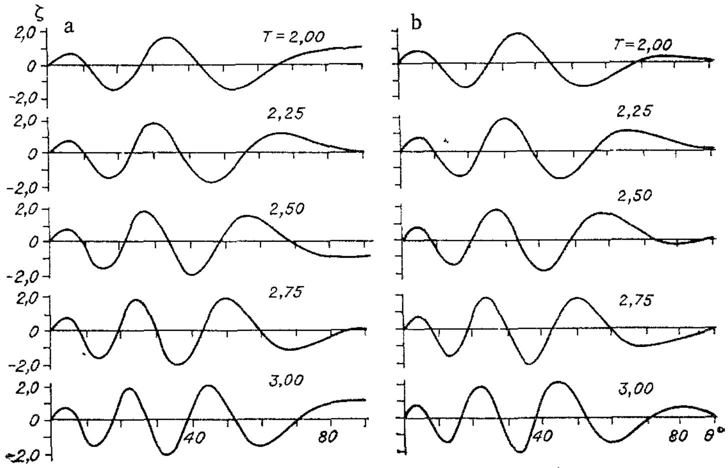

Fig. 4

source action. The conical, constant-phase, surfaces of the unsteady waves intersect this sphere along latitude circles. There is no change in the wave along each latitude, i.e., the wave field is a function only of one coordinate. This method of wave field examination was proposed in [9] in a study of a kindred problem on the forced inertial oscillations in an infinite homogeneous fluid that rotates uniformly relative to the vertical axis.

4. For a numerical computation of the integral representation (2.5) and its asymptotic on an electronic computer, $R = 1$ and $Q = 4\pi$ was taken (such a normalization yields the value 1 for the standing oscillation amplitude at the poles of the unit sphere), and the substitution $z = \cos\theta$, $r = \sin\theta$ is made, where θ is the latitude of the point of observation on the unit sphere. The results of the numerical computation are printed digitally and plotted. They are displayed in Figs. 1-6 as a sweep over four meridians. The Waisalla period $T = 2\pi/N$ is a natural time unit.

The results of the numerical analysis show that the FIW field evolves as follows on the unit sphere. Standing oscillations at the Waisalla frequency are performed at the sphere poles (and only there). At the time when the wave level there becomes maximal, separation of the next wave crest from the pole occurs. The oscillation at the pole reaches the lower position after $\frac{1}{2}T$, and at this time a new wave trough is separated out. The wave being separated moves to the equator, first rapidly and then slowly. Figure 1a shows the evolution of a FIW in the initial stage of 0.25T duration. It is seen that even after the first 0.15T of time the first wave crest has traversed half the path from the pole to the equator of the unit sphere. After an interval of 0.25T measured from the beginning of the wave regime origination, almost $\frac{2}{3}$ the path to the equation has been traversed by this crest. Then the wave motion is abruptly retarded. The wave is shortened and its amplitude grows. Figure 2a shows that in the next $\frac{1}{4}$ Waisalla period the wave will have traversed just barely more than 0.1 of the total path. Later (see Figs. 3a-6a), the wave motion is retarded still more

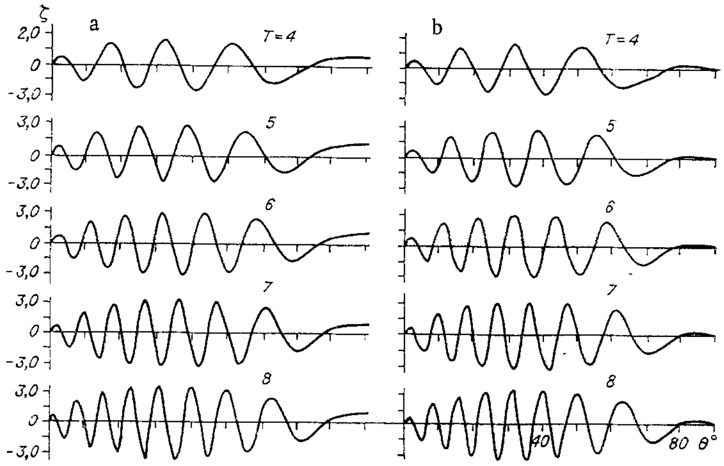

Fig. 5

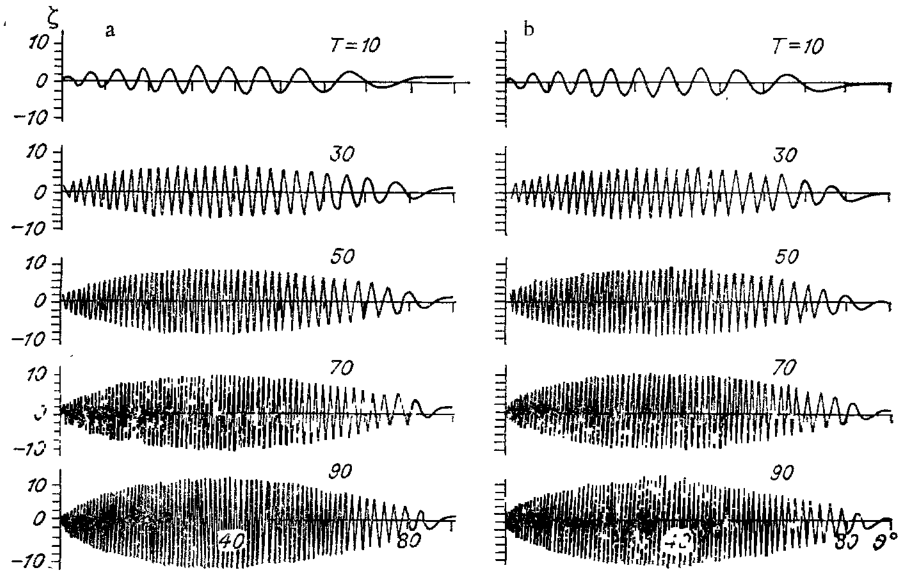

Fig. 6

rapidly. The change in the wave field in the subsequent ¼ Waisalla period is displayed in Fig. 3a. Processes of second wave crest at $t = T$ and second trough at $t = 1.5T$ formation are seen. Furthermore (see Figs. 4a–6a) new FIW peaks and valleys appear after the very same time interval, which equals the Waisalla period. Therefore, as time passes the number of waves appearing in the hemisphere equals the number of Waisalla periods that have passed since the beginning of the oscillatory motion.

The results of computations of the FIW field performed by the asymptotic formula (2.7) are represented in Figs. 1b–6b). It is seen directly that in the domain $20° < \theta < 45°$ good qualitative, and satisfactory quantitative correspondence holds between the exact representation of the FIW field and its asymptotic even for a time on the order of a quarter of a Waisalla period. Later the range of action of the asymptotic formula is extended primarily in the direction to the equator of the unit sphere (following the first wave crest). This formula even acts with high accuracy (on the order of 5% or less) in the zone $0 < \theta < 50°$ for $t = T$. In practice, (2.7) correctly indicates the location of the maximums, minimums, and zeroes of the wave, with the exception of the zero lying to the right of the latitude $\theta = 50°$. For $t = 2T$ the domain of action of the asymptotic is extended to the right to the latitude $\theta = 60°$. For $t = 5T$ the upper boundary of this domain reaches $\theta = 70°$. Further displacement of the quantity θ_M (the upper boundary of the range of action of the asymptotic formula) proceeds slowly: $\theta_M = 77°$ for $t = 10T$, $\theta_M = 82°$ for $t = 30T$, $\theta_M = 84°$ for $t = 50T$, $\theta_M = 85°$ for $t = 70T$, etc.

Therefore, the asymptotic formulas for the FFIWO and the CPSF satisfactorily describe the wave field behavior even in the initial stage of its evolution, with the exception of the domain near the pole of the unit sphere. The wave field is constructed still more simply at the unit sphere poles themselves, as (2.8) shows. The deductions made are also carried over without difficulty to the case of a rotating uniformly stratified fluid. For this case the FFIWO is obtained in [3, 4].

The authors are grateful to V. A. Borovikov and É. V. Teodorovich for interesting interviews related to the topic of this paper.

LITERATURE CITED

1. Yu. Z. Miropol'skii, "Self-similar solutions of the Cauchy problem for interal waves in an unbounded fluid," Izv. Akad. Nauk SSSR, Fizika Atmos. Okeana, 14, No. 9 (1978).
2. S. Ya. Sekerzh-Zen'kovich, "Fundamental solution of the internal wave operator," Dokl. Akad. Nauk SSSR, 246, No. 2 (1979).
3. É. V. Teodorovich and V. A. Gorodtsov, "On certain singular solutions of the internal wave equation," Izv. Akad. Nauk SSSR, Fizika Atmos. Okeana, 16, No. 7 (1980).
4. V. A. Gorodtsov and É. V. Teodorovich, "Cerenkov radiation of internal waves by uniformly moving sources," Preprint No. 183, Inst. Probl. Mekhan. Akad. Nauk SSSR, Moscow (1981).
5. V. S. Vladimirov, Equations of Mathematical Physics [in Russian], Nauka, Moscow (1971).
6. S. Ya. Sekerzh-Zen'kovich, "Uniqueness theorem and explicit representation of the solution of the Cauchy problem for the internal wave equation," Dokl. Akad. Nauk SSSR, 256, No. 2 (1971).
7. S. Ya. Sekerzh-Zen'kovich, "Cauchy problem for the internal wave equation," Prikl. Mat. Mekh., 46, No. 6 (1982).
8. E. T. Copson, Asymptotic Expansions [Russian translation], Mir, Moscow (1966).
9. S. L. Sobolev, "On a new problem of mathematical physics," Izv. Akad. Nauk SSSR, Ser. Mat., 18, No. 1 (1954).

INSTABILITY WAVE EXCITATION BY A LOCALIZED VIBRATOR IN THE BOUNDARY LAYER

A. M. Tumin and A. V. Fedorov

UDC 532.526.013

Modern methods of determining the critical Reynolds numbers for the laminar-turbulent boundary layer transition are based on computations of the linear stage of instability wave development (Tollmien–Schlichting waves), whose initial amplitudes are determined empirically [1, 2]. An analysis of the possible excitation mechanism for Tollmien–Schlichting (T–S) waves by external perturbations is needed to construct closed algorithms to compute the development of these waves. On the other hand, the problem of boundary layer susceptibility to external effects is closely associated with questions of flow laminarization on aircraft vehicles.

It is well known that spatially localized, external perturbations that are periodic in time excite T–S waves effectively [1-3]. Under real flight conditions and in wind tunnel tests the source of such perturbations might be the vibration of the surface being streamlined. Experiments in [4, 5] indicate the close relationship between the origination of instability and the characteristics of model vibration.

A theoretical analysis of the perturbations excited by vibrations of a surface being streamlined in the boundary layer was developed in [6-11]. The first mathematical model related to T–S wave generation by a localized vibrator was constructed by Gaster [6]. A vibrator in sub- and supersonic boundary layers was examined in [7-9]. The analysis was performed within the framework of a three-layer asymptotic model under the assumption that the characteristic scales of the problem correspond to the neighborhood of the lower branch of the neutral curve. Asymptotic expressions are obtained for the longwave pressure perturbations excited in the neighborhood of the vibrating section of the surface. In particular, the amplitude of the damped T–S wave is determined. However, the problem of excitation of growing instability waves emerged beyond the framework of the mathematical model used. In this connection, a postulate was proposed in [10] that permits determination of the amplitude

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 65-72, November-December, 1984. Original article submitted October 5, 1983.

